LieAlgDB

A database of Lie algebras

Version 1.0

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Willem de Graaf
Csaba Schneider

Willem de Graaf — Email: degraaf@science.unitn.it
— Homepage: http://www.science.unitn.it/~degraaf/
— Address: Dipartimento di Matematica
Via Sommarive 14
I-38050 Povo (Trento)
Italy

Csaba Schneider — Email: csaba.schneider@sztaki.hu
— Homepage: http://www.sztaki.hu/~schneider
— Address: Informatics Research Laboratory
Computer and Automation Research Institute
1518 Budapest Pf. 63.
Hungary

Abstract

This package provides access to the the classification of the following families of Lie algebras:

- solvable Lie algebras of dimension at most 4;
- nilpotent Lie algebras of dimension at most 5, and those of dimension 6 over a field of characteristic different from 2;
- non-solvable Lie algebras of dimension at most 6 over finite fields;
- simple Lie algebras of dimension at most 6 over finite fields and those of dimension at most 9 over `GF(2)`.

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Contents

Chapter 1

Introduction

This is the manual for the GAP package LieAlgDB, for accessing and working with several classifications of Lie algebras.

In the mathematics literature many classifications of Lie algebras of various types have been published (we refer to the bibliography for a few examples). However, working with these classifications from paper is not always easy. This package aims at making a few classifications of small dimensional Lie algebras that have appeared in recent years more accessible. For each classification that is contained in the package, functions are provided that construct Lie algebras from that classification inside GAP. This allows the user to obtain easy access to the often rather complicated data contained in a classification, and to directly interface the Lie algebras to the functionality for Lie algebras which is already contained in GAP.

The package contains the following classifications:

- non-solvable Lie algebras over finite fields up to dimension 6 (from [Str]),
- nilpotent Lie algebras of dimension upto 9 over `F=GF(2)`, of dimension upto 7 over `F=GF(3)` or `F=GF(5)` (from [Sch05]),
- simple Lie algebras of dimensions between 7 and 9 over `F=GF(2)` (from [VL06]),
- the classification of solvable Lie algebras of dimensions 2,3, and 4 (from [Gra05]),
- the classification of nilpotent Lie algebras of dimensions 5 and 6 (from [Gra07]).

This manual is structured as follows. The next chapter contains a description of the main functions of the package. Subsequently, the appendices contain descriptions of the classifications used in the package. These are of course also contained in published papers, but for the convenience of the user they have been added here.

Chapter 2

The families of Lie algebras included in the database

2.1 Non-solvable Lie algebras

The package contains the list of non-solvable Lie algebras over finite fields up to dimension 6. The classification follows the one in [Str].

2.1.1 NonSolvableLieAlgebra

◊ `NonSolvableLieAlgebra( F, pars )` (method)

`F` is a finite field, `pars` is a list of parameters with length between 1 and 4. The first entry of `pars` is the dimension of the algebra, and the possible additional entries of `pars` describe the algebra if there are more algebras with dimension `pars[1]`.

The possible values of pars are as follows.

2.1.2 Dimension 1 and 2

There are no non-solvable Lie algebras with dimension less than 3, and so if `pars[1]` is less than 3 then `NonSolvableLieAlgebra` returns an error message.

2.1.3 Dimension 3

There is just 1 non-solvable Lie algebra over an arbitrary finite field `F` (see Section B.1) which is returned by `NonSolvableLieAlgebra( F, [3] )`.

2.1.4 Dimension 4

If `F` has odd characteristic then there is a unique non-solvable Lie algebra with dimension 4 over `F` and this algebra is returned by `NonSolvableLieAlgebra( F, [4] )`. If `F` has characteristic 2, then there are two distinct Lie algebras and they are returned by `NonSolvableLieAlgebra( F, [4,i] )` for `i=1, 2`. See Section B.2 for a description of the algebras.

2.1.5 Dimension 5

If `F` has characteristic 2 then there are 5 isomorphism classes of non-solvable Lie algebras over `F` and they are described in Sections B.3.1. The possible values of `pars` are as follows.

- `[5,1]`: the Lie algebra in B.3.1(1).
- `[5,2,a]`: `a=0, 1`; the Lie algebras in B.3.1(2).
- `[5,3,a]`: `a=0, 1`; the Lie algebras in B.3.1(3).

If the characteristic of `F` is odd, then the list of Lie algebras is as follows (see Section B.3.2).

- `[5,1,a]`: `a=1, 0`; the Lie algebras that occur in B.3.2(1).
- `[5,2]`: the Lie algebra in B.3.2(2).
- `[5,3]`: this algebra only exists if the characteristic of `F` is 3 or 5. In the former case the algebra is the one in B.3.2(3), while in the latter it is in B.3.2(4).

2.1.6 Dimension 6

The 6-dimensional non-solvable Lie algebras are described in Section B.4. If `F` has characteristic 2, then the possible values of `pars` is as follows.

- `[6,1]`: the Lie algebra in B.4.1(1).
- `[6,2]`: the Lie algebra in B.4.1(2).
- `[6,3,a]`: `a=0, 1`; the two Lie algebras B.4.1(3).
- `[6,4,a]`: `a=0, 1, 2, 3` or `a` is a field element. In this case `NonSolvableLieAlgebras` returns one of the Lie algebras in B.4.1(4). If `a=0, 1, 2, 3` then the Lie algebra corresponding to the `a+1`-th matrix of B.4.1(4) is returned. If `a` is a field element, then a Lie algebra is returned which corresponds to the 4th matrix in B.4.1(4).
- `[6,5]`: the Lie algebra in B.4.1(5).
- `[6,6,1], [6,6,2], [6,6,3,a], [6,6,4,a]: a` is a field element; the Lie algebras in B.4.1(6). The third and fourth entries of `pars` determine the isomorphism type of the radical as a solvable Lie algebra. More precisely, if the third argument `pars[3]` is 1 or 2 then the radical is isomorphic to `SolvableLieAlgebra( F,[3,pars[3]] )`. If the third argument `pars[3]` is 3 or 4 then the radical is isomorphic to `SolvableLieAlgebra( F,[3,pars[3],pars[4]] )`; see `SolvableLieAlgebra` (2.2.1).
- `[6,7]`: the Lie algebra in B.4.1(7).
- `[6,8]`: the Lie algebra in B.4.1(8).

If the characteristic of `F` is odd, then the possible values of `pars` are the following (see Sections B.4.2, B.4.3, and B.4.4).

- `[6,1]`: the Lie algebra in B.4.2(1).

- `[6,2]`: the Lie algebra in B.4.2(2).
- `[6,3,1], [6,3,2], [6,3,3,a], [6,3,4,a]: a` is a field element; the Lie algebras in B.4.2(3). The third and fourth entries of `pars` determine the isomorphism type of the radical as a solvable Lie algebra. More precisely, if the third argument `pars[3]` is 1 or 2 then the radical is isomorphic to `SolvableLieAlgebra( F,[3,pars[3]] )`. If the third argument `pars[3]` is 3 or 4 then the radical is isomorphic to `SolvableLieAlgebra( F,[3,pars[3],pars[4]] )`; see `SolvableLieAlgebra` (2.2.1).
- `[6,4]`: the Lie algebra in B.4.2(4).
- `[6,5]`: the Lie algebra in B.4.2(5).
- `[6,6]`: the Lie algebra in B.4.2(6).
- `[6,7]`: the Lie algebra in B.4.2(7).

If the characteristic is 3 or 5 then there are additional families. In characteristic 3, these families are as follows.

- `[6,8,a]`: `a` is a field element; returns one of the Lie algebras in B.4.3(1).
- `[6,9]`: the Lie algebra in B.4.3(2).
- `[6,10]`: the Lie algebra in B.4.3(3).
- `[6,11,a]`: `a=0, 1`; one of the two Lie algebras in B.4.3(4).
- `[6,12]`: the first Lie algebra in B.4.3(5).
- `[6,13]`: the second Lie algebra B.4.3(5).

If the characteristic is 5, then the additional Lie algebras are the following.

- `[6,8]`: the Lie algebra in B.4.4(1).
- `[6,9]`: the Lie algebra in B.4.4(2).

2.1.7 NonSolvableLieAlgebras

◊ `NonSolvableLieAlgebras( F, dim )` (method)

Here `F` is a finite field, and `dim` is at least 6. The list of non-solvable Lie algebras over `F` of dimension `dim` is returned.

2.1.8 SimpleLieAlgebras

◊ `SimpleLieAlgebras( F, dim )` (method)

Here `F` is a finite field, and `dim` is either an integer not greater than 6, or, if `F=GF(2)`, then `dim` is not greater than 9. The output is the list of simple Lie algebras over `F` of dimension `dim`. If `dim` is at most 6, then the classification by Strade [Str] is used. If `F=GF(2)` and `dim` is between 7 and 9, then the Lie algebras in [VL06] are returned.

2.2 Solvable and nilpotent Lie algebras

The package contains the classification of solvable Lie algebras of dimensions 2,3, and 4 (taken from [Gra05]), and the classification of nilpotent Lie algebras of dimensions 5 and 6 (from [Gra07]). The classification of nilpotent Lie algebras of dimension 6 is only complete for base fields of characteristic not 2. The classifications are complemented by a function for identifying a given Lie algebra as a member of the list. This function also returns an explicit isomorphism. In Appendix A the list is given of the multiplication tables of the solvable and nilpotent Lie algebras, corresponding to the functions in this section.

2.2.1 SolvableLieAlgebra

◊ `SolvableLieAlgebra( F, pars )` (method)

Here `F` is a field, `pars` is a list of parameters with length between `2` and `4`. The first entry of `pars` is the dimension of the algebra, which has to be 2, 3, or 4. If the dimension is 3, or 4, then the second entry of `pars` is the number of the Lie algebra with which it appears in the list of [Gra05]. If the dimension is 2, then there are only two (isomorphism classes of) solvable Lie algebras. In this case, if the second entry is 1, then the abelian Lie algebra is returned, if it is 2, then the unique non-abelian solvable Lie algebra of dimension 2 is returned. A Lie algebra in the list of [Gra05] can have one or two parameters. In that case the list `pars` also has to contain the parameters.

Example
```
gap> SolvableLieAlgebra( Rationals, [4,6,1,2] );
<Lie algebra of dimension 4 over Rationals>
```

2.2.2 NilpotentLieAlgebra

◊ `NilpotentLieAlgebra( F, pars )` (method)

Here `F` is a field, `pars` is a list of parameters with length between `2` and `3`. The first entry of `pars` is the dimension of the algebra, which has to be 5 or 6. The second entry of `pars` is the number of the Lie algebra with which it appears in the list of [Gra07]. A Lie algebra in the list of [Gra07] can have one parameter. In that case the list `pars` also has to contain the parameter.

Example
```
gap> NilpotentLieAlgebra( GF(3^7), [ 6, 24, Z(3^7)^101 ] );
<Lie algebra of dimension 6 over GF(3^7)>
```

2.2.3 SolvableLieAlgebras

◊ `SolvableLieAlgebras( F, dim )` (method)

Here `F` is a finite field, and `dim` is one of 2,3,4. The list of all solvable Lie algebras over `F` of dimension `dim` is returned.

2.2.4 NilpotentLieAlgebras

◊ `NilpotentLieAlgebras( F, dim )` (method)

Here F is a finite field, and dim not greater than 9 if F=GF(2), dim is not greater than 7 if F=GF(3) or F=GF(5), and dim is not greater than 6 otherwise. The list of all nilpotent Lie algebras over F of dimension dim is returned. If dim is not greater than 6, then the list of nilpotent Lie algebras is determined by [Gra07], otherwise the classification can be found in [Sch05].

2.2.5 NumberOfNilpotentLieAlgebras

◊ NumberOfNilpotentLieAlgebras(F, dim) (method)

Here F is a finite field, and dim not greater than 9 if F=GF(2), dim is not greater than 7 if F=GF(3) or F=GF(5), and dim is not greater than 6 otherwise. The number of nilpotent Lie algebras over F of dimension dim is returned.

2.2.6 LieAlgebraIdentification

◊ LieAlgebraIdentification(L) (method)

Here L is a solvable Lie algebra of dimension 2,3, or 4, or it is a nilpotent Lie algebra of dimension 5 or 6 (in the latter case it has to be of characteristic not 2). This function returns a record with three fields.

- name This is a string containing the name of the Lie algebra. It starts with a capital L if it is a solvable Lie algebra of dimension 2,3,4. It starts with a capital N if it is a nilpotent Lie algebra of dimension 5 or 6. A name like

  ```
  "N6_24( GF(3^7), Z(3^7) )"
  ```

  means that the input Lie algebra is isomorphic to the Lie algebra with number 24 in the list of 6-dimensional nilpotent Lie algebras. Furthermore the field is given and the value of the parameters (if there are any).

- parameters This contains the parameters that appear in the name of the algebra.

- isomorphism This is an isomorphism of the input Lie algebra to the Lie algebra from the classification with the given name.

Example
```
gap> L:= SolvableLieAlgebra( Rationals, [4,7,1/2,4/9] );
<Lie algebra of dimension 4 over Rationals>
gap> LieAlgebraIdentification( L );
rec( name := "L4_7( Rationals, 256/729, 256/729 )",
  parameters := [ 256/729, 256/729 ],
  isomorphism := CanonicalBasis( <Lie algebra of dimension
    4 over Rationals> ) ->
    [ (10)*v.1+v.2+(4)*v.3, (16/9)*v.1+(848/81)*v.2+(8/9)*v.3,
      (32/81)*v.1+(1408/729)*v.2+(6784/729)*v.3, (8/9)*v.4 ] )
```

In the example we see that the program finds different parameters, than the ones with which the Lie algebra was constructed. The explanation is that some parametric classes of Lie algebras contain isomorphic Lie algebras, for different values of the parameters. In that case the identification function makes its own choice.

Appendix A

Description of the solvable and nilpotent Lie algebras

In this appendix we list the multiplication tables of the nilpotent and solvable Lie algebras contained in the package. Some parametric classes contain isomorphic Lie algebras, for different values of the parameters. For exact descriptions of these isomorphisms we refer to [Gra05], [Gra07]. In dimension 2 there are just two classes of solvable Lie algebras:

- L_2^1: The Abelian Lie algebra.
- L_2^2: $[x_2,x_1] = x_1$.

We have the following solvable Lie algebras of dimension 3:

- L_3^1 The Abelian Lie algebra.
- L_3^2 $[x_3,x_1] = x_1, [x_3,x_2] = x_2$.
- $L_3^3(a)$ $[x_3,x_1] = x_2, [x_3,x_2] = ax_1 + x_2$.
- $L_3^4(a)$ $[x_3,x_1] = x_2, [x_3,x_2] = ax_1$.

And the following solvable Lie algebras of dimension 4:

- L_4^1 The Abelian Lie algebra.
- L_4^2 $[x_4,x_1] = x_1, [x_4,x_2] = x_2, [x_4,x_3] = x_3$.
- $L_4^3(a)$ $[x_4,x_1] = x_1, [x_4,x_2] = x_3, [x_4,x_3] = -ax_2 + (a+1)x_3$.
- L_4^4 $[x_4,x_2] = x_3, [x_4,x_3] = x_3$.
- L_4^5 $[x_4,x_2] = x_3$.
- $L_4^6(a,b)$ $[x_4,x_1] = x_2, [x_4,x_2] = x_3, [x_4,x_3] = ax_1 + bx_2 + x_3$.
- $L_4^7(a,b)$ $[x_4,x_1] = x_2, [x_4,x_2] = x_3, [x_4,x_3] = ax_1 + bx_2$.
- L_4^8 $[x_1,x_2] = x_2, [x_3,x_4] = x_4$.
- $L_4^9(a)$ $[x_4,x_1] = x_1 + ax_2, [x_4,x_2] = x_1, [x_3,x_1] = x_1, [x_3,x_2] = x_2$.

- $L_4^{10}(a)$ $[x_4,x_1]=x_2,[x_4,x_2]=ax_1,[x_3,x_1]=x_1,[x_3,x_2]=x_2$.
- $L_4^{11}(a,b)$ $[x_4,x_1]=x_1,[x_4,x_2]=bx_2,[x_4,x_3]=(1+b)x_3,[x_3,x_1]=x_2,[x_3,x_2]=ax_1$. Condition on F: the characteristic of F is 2.
- L_4^{12} $[x_4,x_1]=x_1,[x_4,x_2]=2x_2,[x_4,x_3]=x_3,[x_3,x_1]=x_2$.
- $L_4^{13}(a)$ $[x_4,x_1]=x_1+ax_3,[x_4,x_2]=x_2,[x_4,x_3]=x_1,[x_3,x_1]=x_2$.
- $L_4^{14}(a)$ $[x_4,x_1]=ax_3,[x_4,x_3]=x_1,[x_3,x_1]=x_2$.

Nilpotent of dimension 5:

- $N_{5,1}$ Abelian.
- $N_{5,2}$ $[x_1,x_2]=x_3$.
- $N_{5,3}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4$.
- $N_{5,4}$ $[x_1,x_2]=x_5,[x_3,x_4]=x_5$.
- $N_{5,5}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_5,[x_2,x_4]=x_5$.
- $N_{5,6}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_2,x_3]=x_5$.
- $N_{5,7}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5$.
- $N_{5,8}$ $[x_1,x_2]=x_4,[x_1,x_3]=x_5$.
- $N_{5,9}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5$.

We get nine 6-dimensional nilpotent Lie algebras denoted $N_{6,k}$ for k=1,...,9 that are the direct sum of $N_{5,k}$ and a 1-dimensional abelian ideal. Subsequently we get the following Lie algebras.

- $N_{6,10}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_6,[x_4,x_5]=x_6$.
- $N_{6,11}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_6,[x_2,x_3]=x_6,[x_2,x_5]=x_6$.
- $N_{6,12}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_6,[x_2,x_5]=x_6$.
- $N_{6,13}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_5,[x_2,x_4]=x_5,[x_1,x_5]=x_6,[x_3,x_4]=x_6$.
- $N_{6,14}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_2,x_3]=x_5,[x_2,x_5]=x_6,[x_3,x_4]=-x_6$.
- $N_{6,15}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_2,x_3]=x_5,[x_1,x_5]=x_6,[x_2,x_4]=x_6$.
- $N_{6,16}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_2,x_5]=x_6,[x_3,x_4]=-x_6$.
- $N_{6,17}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_1,x_5]=x_6,[x_2,x_3]=x_6$.
- $N_{6,18}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_1,x_4]=x_5,[x_1,x_5]=x_6$.
- $N_{6,19}(a)$ $[x_1,x_2]=x_4,[x_1,x_3]=x_5,[x_2,x_4]=x_6,[x_3,x_5]=ax_6$.
- $N_{6,20}$ $[x_1,x_2]=x_4,[x_1,x_3]=x_5,[x_1,x_5]=x_6,[x_2,x_4]=x_6$.

- $N_{6,21}(a)$ $[x_1,x_2]=x_3,[x_1,x_3]=x_4,[x_2,x_3]=x_5,[x_1,x_4]=x_6,[x_2,x_5]=ax_6$.
- $N_{6,22}(a)$ $[x_1,x_2]=x_5,[x_1,x_3]=x_6,[x_2,x_4]=ax_6,[x_3,x_4]=x_5$.
- $N_{6,23}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_5,[x_1,x_4]=x_6,[x_2,x_4]=x_5$.
- $N_{6,24}(a)$ $[x_1,x_2]=x_3,[x_1,x_3]=x_5,[x_1,x_4]=ax_6,[x_2,x_3]=x_6,[x_2,x_4]=x_5$.
- $N_{6,25}$ $[x_1,x_2]=x_3,[x_1,x_3]=x_5,[x_1,x_4]=x_6$.
- $N_{6,26}$ $[x_1,x_2]=x_4,[x_1,x_3]=x_5,[x_2,x_3]=x_6$.

Appendix B

Description of the non-solvable Lie algebras

In this appendix we list the non-solvable Lie algebras contained in the package. Our notation follows [Str], where a more detailed description can also be found. In particular if L is a Lie algebra over F then $C(L)$ denotes the center of L. Further, if $x_1,\dots,x_k$ are elements of L, then $F<x_1,\dots,x_k>$ denotes the linear subspace generated by $x_1,\dots,x_k$, and we also write Fx_1 for $F<x_1>$

B.1 Dimension 3

There are no non-solvable Lie algebras with dimension 1 or 2. Over an arbitrary finite field `F`, there is just one isomorphism type of non-solvable Lie algebras:

1. If `char F=2` then the algebra is $W(1;\underline{2})^{(1)}$.
2. If `char F>2` then the algebra is $\mathrm{sl}(2,F)$.

See Theorem 3.2 of [Str] for details.

B.2 Dimension 4

Over a finite field `F` of characteristic 2 there are two isomorphism classes of non-solvable Lie algebras with dimension 4, while over a finite field `F` of odd characteristic the number of isomorphism classes is one (see Theorem 4.1 of [Str]). The classes are as follows:

1. characteristic 2: $W(1;\underline{2})$ and $W(1;\underline{2})^{(1)}\oplus F$.
2. odd characteristic: $\mathrm{gl}(2,F)$.

B.3 Dimension 5

B.3.1 Characteristic 2

Over a finite field `F` of characteristic 2 there are 5 isomorphism classes of non-solvable Lie algebras with dimension 5:

1. $\mathrm{Der}(W(1;\underline{2})^{(1)})$;
2. $W(1;\underline{2}) \ltimes Fu$ where $[W(1;\underline{2})^{(1)},u]=0$, $[x^{(3)}\partial,u]=\delta u$ and $\delta\in\{0,1\}$ (two algebras);
3. $W(1;\underline{2})^{(1)} \oplus (F<h,u>)$, $[h,u]=\delta u$, where $\delta\in\{0,1\}$ (two algebras).

See Theorem 4.2 of [Str] for details.

B.3.2 Odd characteristic

Over a field $F$of odd characteristic the number of isomorphism types of 5-dimensional non-solvable Lie algebras is 3 if the characteristic is at least 7, and it is 4 otherwise (see Theorem 4.3 of [Str]). The classes are as follows.

1. $\mathrm{sl}(2,F)\oplus F<x,y>$, $[x,y]=\delta y$ where $\delta\in\{0,1\}$.
2. $\mathrm{sl}(2,F)\ltimes V(1)$ where $V(1)$ is the irreducible 2-dimensional $\mathrm{sl}(2,F)$=module.
3. If char $F=3$ then there is an additional algebra, namely the non-split extension $0\to V(1)\to L\to \mathrm{sl}(2,F)\to 0$.
4. If char $F=5$ then there is an additional algebra: $W(1;\underline{1})$.

B.4 Dimension 6

B.4.1 Characteristic 2

Over a field F of characteristic 2, the isomorphism classes of non-solvable Lie algebras are as follows.

1. $W(1;\underline{2})^{(1)}\oplus W(1;\underline{2})^{(1)}$.
2. $W(1;\underline{2})^{(1)}\otimes F_{q^2}$ where $F=F_q$.
3. $\mathrm{Der}(W(1;\underline{2})^{(1)})\ltimes Fu$, $[W(1;\underline{2}),u]=0$, $[\partial^2,u]=\delta u$ where $\delta=\{0,1\}$.
4. $W(1;\underline{2})\ltimes(F<h,u>)$, $[W(1;\underline{2})^{(1)},(F<h,u>]=0$, $[h,u]=\delta u$, and if $\delta=0$, then the action of $x^{(3)}\partial$ on $F<h,u>$ is given by one of the following matrices:
$$\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}0&\xi\\1&1\end{pmatrix}\text{ where }\xi\in F^*.$$
5. the algebra is as in (4.), but $\delta=1$. Note that Theorem 5.1(3/b) of [Str] lists two such algebras but they turn out to be isomorphic. We take the one with $[x^{(3)}\partial,h]=[x^{(3)}\partial,u]=0$.
6. $W(1;\underline{2})^{(1)}\oplus K$ where K is a 3-dimensional solvable Lie algebra.
7. $W(1;\underline{2})^{(1)}\ltimes O(1;\underline{2})/F$.
8. the non-split extension $0\to O(1;\underline{2})/F\to L\to W(1;\underline{2})^{(1)}\to 0$.

See Theorem 5.1 of [Str].

B.4.2 General odd characteristic

If the characteristic of the field is odd, then the 6-dimensional non-solvable Lie algebras are described by Theorem 5.2–5.4 of [Str]. Over such a field F, let us define the following isomorphism classes of 6-dimensional non-solvable Lie algebras.

1. $\mathrm{sl}(2,F) \oplus \mathrm{sl}(2,F)$.
2. $\mathrm{sl}(2,F_{q^2})$ where $F = F_q$;
3. $\mathrm{sl}(2,F) \oplus K$ where K is a solvable Lie algebra with dimension 3;
4. $\mathrm{sl}(2,F) \ltimes (V(0) \oplus V(1))$ where $V(i)$ is the $(i+1)$-dimensional irreducible $\mathrm{sl}(2,F)$-module;
5. $\mathrm{sl}(2,F) \ltimes V(2)$ where $V(2)$ is the 3-dimensional irreducible $\mathrm{sl}(2,F)$-module;
6. $\mathrm{sl}(2,F) \ltimes (V(1) \oplus C(L)) \cong \mathrm{sl}(2,F) \ltimes H$ where H is the Heisenberg Lie algebra;
7. $\mathrm{sl}(2,F) \ltimes K$ where $K = Fd \oplus K^{(1)}$, $K^{(1)}$ is 2-dimensional abelian, isomorphic, as an $\mathrm{sl}(2,F)$-module, to $V(1)$, $[\mathrm{sl}(2,F),d] = 0$, and, for all $v \in K$, $[d,v] = v$;

If the characteristic of F is at least 7, then these algebras form a complete and irredundant list of the isomorphism classes of the 6-dimensional non-solvable Lie algebras.

B.4.3 Characteristic 3

If the characteristic of the field F is 3, then, besides the classes in Section B.4.2, we also obtain the following isomorphism classes.

1. $\mathrm{sl}(2,F) \ltimes V(2,\chi)$ where χ is a 3-dimensional character of $\mathrm{sl}(2,F)$. Each such character is described by a field element a such that $T^3 + T^2 - a$ has a root in F; see Proposition 3.5 of [Str] for more details.
2. $W(1;\underline{1}) \ltimes O(1;\underline{1})$ where $O(1;\underline{1})$ is considered as an abelian Lie algebra.
3. $W(1;\underline{1}) \ltimes O(1;\underline{1})^*$ where $O(1;\underline{1})^*$ is the dual of $O(1;\underline{1})$ and it is considered as an abelian Lie algebra.
4. One of the two 6-dimensional central extensions of the non-split extension $0 \to V(1) \to L \to \mathrm{sl}(2,F) \to 0$; see Proposition 4.5 of [Str]. We note that Proposition 4.5 of [Str] lists three such central extensions, but one of them is not a Lie algebra.
5. One of the two non-split extensions $0 \to \mathrm{rad}\, L \to L \to L/\mathrm{rad}\, L \to 0$ with a 5-dimensional ideal; see Theorem 5.4 of [Str].

We note here that [Str] lists one more non-solvable Lie algebra over a field of characteristic 3, namely the one in Theorem 5.3(5). However, this algebra is isomorphic to the one in Theorem 5.3(4).

B.4.4 Characteristic 5

If the characteristic of the field F is 5, then, besides the classes in Section B.4.2, we also obtain the following isomorphism classes.

1. $W(1;\underline{1}) \oplus F$.
2. The non-split central extension $0 \to F \to L \to W(1;\underline{1}) \to 0$.

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