GAP 4 Package ExamplesForHomalg

ExamplesForHomalg — Examples for the GAP package homalg

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(*this manual is still under construction*)

This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~barakat/ExamplesForHomalg/chap0.html

An OFFLINE version should be included in the documentation subfolder of the package. This package is part of the homalg-project:

http://homalg.math.rwth-aachen.de/index.php/core-packages/examplesforhomalg

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Contents

Chapter 1

Introduction

[Bar09]

Chapter 2

Installation of the ExamplesForHomalg Package

To install this package just extract the package's archive file to the GAP `pkg` directory.

By default the ExamplesForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("ExamplesForHomalg");
```

before its functions become available.

Please, send us an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat and Simon Görtzen.

Chapter 3

Examples

3.1 Spectral Filtrations

3.1.1 ExtExt

This is Example B.2 in [Bar].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A homalg external 6 by 5 matrix>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> Y := Hom( Qxyz, W );
<A right module on 5 generators satisfying yet unknown relations>
gap> F := InsertObjectInMultiFunctor( Functor_Hom, 2, Y, "TensorY" );
<The functor TensorY>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
```

```
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 * s s s
 . . s *
 . . . *
---------
Level 4:

 s s s s
 . s s s
```

```
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0:	<A non-zero left module presented by 33 relations for 23 generators>
  -1:	<A non-zero left module presented by 37 relations for 22 generators>
  -2:	<A non-zero left module presented by 20 relations for 8 generators>
  -3:	<A non-zero left module presented by 29 relations for 4 generators>
of
<A non-zero left module presented by 111 relations for 37 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0:	<A non-zero left module presented by 25 relations for 16 generators>
  -1:	<A non-zero left module presented by 30 relations for 14 generators>
  -2:	<A non-zero left module presented by 18 relations for 7 generators>
  -3:	<A non-zero left module presented by 12 relations for 4 generators>
of
<A non-zero left module presented by 48 relations for 20 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

3.1.2 Purity

This is Example B.3 in [Bar].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A homalg external 6 by 5 matrix>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0:	<A codegree-[ 1, 1 ]-pure rank 2 left module presented by
3 relations for 4 generators>
  -1:	<A codegree-1-pure codim 1 left module presented by 4 relations for
3 generators>
  -2:	<A cyclic reflexively pure codim 2 left module presented by
2 relations for a cyclic generator>
  -3:	<A cyclic reflexively pure codim 3 left module presented by
3 relations for a cyclic generator>
of
<A non-pure rank 2 left module presented by 6 relations for 5 generators>>
gap> W;
<A non-pure rank 2 left module presented by 6 relations for 5 generators>
gap> II_E := SpectralSequence( filt );
```

```
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 s . . .
 * s . .
```

```
 . * * .
 . . . *
---------
Level 3:

 s . . .
 * s . .
 . . s .
 . . . *
---------
Level 4:

 s . . .
 . s . .
 . . s .
 . . . s

gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), W );
true
gap> Source( m );
<A left module presented by 12 relations for 9 generators (locked)>
 gap> Display( last );
 0,  0,   x, -y,0,1, 0,    0,  0,
 x*y,-y*z,-z,0, 0,0, 0,    0,  0,
 x^2,-x*z,0, -z,1,0, 0,    0,  0,
 0,  0,   0, 0, y,-z,0,    0,  0,
 0,  0,   0, 0, x,0, -z,   0,  1,
 0,  0,   0, 0, 0,x, -y,   -1, 0,
 0,  0,   0, 0, 0,-y,x^2-1,0,  0,
 0,  0,   0, 0, 0,0, 0,    z,  0,
 0,  0,   0, 0, 0,0, 0,    y-1,0,
 0,  0,   0, 0, 0,0, 0,    0,  z,
 0,  0,   0, 0, 0,0, 0,    0,  y,
 0,  0,   0, 0, 0,0, 0,    0,  x

 Cokernel of the map

 Q[x,y,z]^(1x12) --> Q[x,y,z]^(1x9),

 currently represented by the above matrix
 gap> Display( filt );
 Degree 0:

 0,  0,   x, -y,
 x*y,-y*z,-z,0,
 x^2,-x*z,0, -z

 Cokernel of the map

 Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x4),
```

```
currently represented by the above matrix
----------
Degree -1:

y,-z,0,
x,0, -z,
0,x, -y,
0,-y,x^2-1

Cokernel of the map

Q[x,y,z]^(1x4) --> Q[x,y,z]^(1x3),

currently represented by the above matrix
----------
Degree -2:

Q[x,y,z]/< z, y-1 >
----------
Degree -3:

Q[x,y,z]/< z, y, x >
gap> Display( m );
1,   0,    0,  0,   0,
0,   -1,   0,  0,   0,
0,   0,    -1, 0,   0,
0,   0,    0,  -1,  0,
-x^2,-x*z, 0,  -z,  0,
0,   0,    x,  -y,  0,
0,   0,    0,  0,   -1,
0,   0,    x^2,-x*y,y,
x^3, x^2*z,0,  x*z, -z

the map is currently represented by the above 9 x 5 matrix
```

3.1.3 A3_Purity

This is Example B.4 in [Bar].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );;
gap> nmat := HomalgMatrix( "[ \
> 3*Dy*Dz-Dz^2+Dx+3*Dy-Dz,           3*Dy*Dz-Dz^2,     \
> Dx*Dz+Dz^2+Dz,                     Dx*Dz+Dz^2,       \
> Dx*Dy,                             0,                \
> Dz^2-Dx+Dz,                        3*Dx*Dy+Dz^2,     \
> Dx^2,                              0,                \
> -Dz^2+Dx-Dz,                       3*Dx^2-Dz^2,      \
> Dz^3-Dx*Dz+Dz^2,                   Dz^3,             \
> 2*x*Dz^2-2*x*Dx+2*x*Dz+3*Dx+3*Dz+3,2*x*Dz^2+3*Dx+3*Dz\
> ]", 8, 2, A3 );
<A homalg external 8 by 2 matrix>
```

```
gap> N := LeftPresentation( nmat );
<A left module presented by 8 relations for 2 generators>
gap> filt := PurityFiltration( N );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0:        <A zero left module>
  -1:        <A cyclic reflexively pure codim 1 left module presented by
1 relation for a cyclic generator>
  -2:        <A cyclic reflexively pure codim 2 left module presented by
2 relations for a cyclic generator>
  -3:        <A cyclic reflexively pure codim 3 left module presented by
3 relations for a cyclic generator>
of
<A non-pure codim 1 left module presented by 8 relations for 2 generators>>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 4 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 4 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * * *
 . * * * *
 . . * * *
 . . . * *
---------
Level 1:

 * * * * *
 . . . . .
 . . . . .
 . . . . .
---------
Level 2:

 s . . . .
 . . . . .
 . . . . .
 . . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 4 ] ]
---------
Level 0:

 * * * *
 * * * *
```

```
 . * * *
 . . * *
 . . . *
---------
Level 1:

 . . * *
 * * * *
 . * * *
 . . * *
 . . . .
---------
Level 2:

 . . . .
 s . . .
 . s . .
 . . s .
 . . . .
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), N );
true
gap> Source( m );
<A left module presented by 6 relations for 3 generators (locked)>
 gap> Display( last );
 Dx,-1/3,-2/9*x,
 0, Dy,  -1/3,
 0, Dx,  1,
 0, 0,   Dz,
 0, 0,   Dy,
 0, 0,   Dx

 Cokernel of the map

 R^(1x6) --> R^(1x3), ( for R := Q[x,y,z]<Dx,Dy,Dz> )

 currently represented by the above matrix
 gap> Display( filt );
 Degree 0:

 0
 ----------
 Degree -1:

 Q[x,y,z]<Dx,Dy,Dz>/< Dx >
 ----------
 Degree -2:

 Q[x,y,z]<Dx,Dy,Dz>/< Dy, Dx >
 ----------
 Degree -3:
```

```
Q[x,y,z]<Dx,Dy,Dz>/< Dz, Dy, Dx >
gap> Display( m );
1,               1,
-3*Dz-3,         -3*Dz,
-3*Dz^2+3*Dx-3*Dz,-3*Dz^2

the map is currently represented by the above 3 x 2 matrix
```

3.1.4 TorExt-Grothendieck

This is Example B.5 in [Bar].

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A homalg external 6 by 5 matrix>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct, 2, W, "TensorW" );
<The functor TensorW>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:
```

```
 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3:        <A non-zero cyclic left module presented by
3 relations for a cyclic generator>
  -2:        <A non-zero left module presented by 17 relations for 6 generators>
  -1:        <A non-zero left module presented by 19 relations for 9 generators>
   0:        <A non-zero left module presented by 13 relations for 10 generators>
of
```

```
<A left module presented by yet unknown relations for 41 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3:        <A non-zero cyclic left module presented by
3 relations for a cyclic generator>
  -2:        <A non-zero left module presented by 12 relations for 4 generators>
  -1:        <A non-zero left module presented by 18 relations for 8 generators>
   0:        <A non-zero left module presented by 11 relations for 10 generators>
of
<A left module presented by 21 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

3.1.5 TorExt

This is Example B.6 in [Bar].

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A homalg external 6 by 5 matrix>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> P := Resolution( W );
<A right acyclic complex containing 3 morphisms of left modules at degrees
[ 0 .. 3 ]>
gap> GP := Hom( P );
<A cocomplex containing 3 morphisms of right modules at degrees [ 0 .. 3 ]>
gap> FGP := GP * P;
<A cocomplex containing 3 morphisms of left complexes at degrees [ 0 .. 3 ]>
gap> BC := HomalgBicomplex( FGP );
<A bicocomplex containing left modules at bidegrees [ 0 .. 3 ]x[ -3 .. 0 ]>
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0 .. 3 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:
```

```
 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:
```

```
 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3:        <A non-zero cyclic left module presented by
3 relations for a cyclic generator>
  -2:        <A non-zero left module presented by 17 relations for 7 generators>
  -1:        <A non-zero left module presented by 25 relations for 12 generators>
   0:        <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by yet unknown relations for 24 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3:        <A non-zero cyclic left module presented by
3 relations for a cyclic generator>
  -2:        <A non-zero left module presented by 12 relations for 4 generators>
  -1:        <A non-zero left module presented by 21 relations for 8 generators>
   0:        <A non-zero left module presented by 11 relations for 10 generators>
of
<A left module presented by 23 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

3.1.6 CodegreeOfPurity

This is Example B.7 in [Bar].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> vmat := HomalgMatrix( "[ \
> 0,   0,   x,-z, \
> x*z,z^2,y,0,   \
> x^2,x*z,0,y    \
> ]", 3, 4, Qxyz );
<A homalg external 3 by 4 matrix>
gap> V := LeftPresentation( vmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> wmat := HomalgMatrix( "[ \
> 0,   0,   x,-y, \
> x*y,y*z,z,0,   \
> x^2,x*z,0,z    \
> ]", 3, 4, Qxyz );
<A homalg external 3 by 4 matrix>
gap> W := LeftPresentation( wmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> Rank( V );
2
gap> Rank( W );
2
gap> ProjectiveDimension( V );
```

```
2
gap> ProjectiveDimension( W );
2
gap> DegreeOfTorsionFreeness( V );
1
gap> DegreeOfTorsionFreeness( W );
1
gap> CodegreeOfPurity( V );
[ 2 ]
gap> CodegreeOfPurity( W );
[ 1, 1 ]
gap> filtV := PurityFiltration( V );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
   0:        <A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for
4 generators>
  -1:        <A zero left module>
  -2:        <A zero left module>
of
<A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for
4 generators>>
gap> filtW := PurityFiltration( W );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
   0:        <A codegree-[ 1, 1 ]-pure rank 2 left module presented by
3 relations for 4 generators>
  -1:        <A zero left module>
  -2:        <A zero left module>
of
<A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for
4 generators>>
gap> II_EV := SpectralSequence( filtV );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EV );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 * * *
 . * *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
```

```
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
---------
Level 1:

 * * * *
 * * * *
 . . * *
---------
Level 2:

 * . . .
 * . . .
 . . * *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
gap> II_EW := SpectralSequence( filtW );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EW );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:
```

```
 * * *
 * * *
 . * *
 . . *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . * * *
 . . . *
---------
Level 2:

 * . . .
 . * . .
 . . . *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
```

```
  . . . s
```

3.1.7 HomHom

This corresponds to the example of Section 2 in [BR06].

```
gap> R := HomalgRingOfIntegersInExternalGAP( ) / 2^8;
<A homalg residue class ring>
gap> Display( R );
Z/( 256 )
gap> M := LeftPresentation( [ 2^5 ], R );
<A cyclic left module presented by an unknown number of relations for a cyclic\
 generator>
gap> Display( M );
Z/( 256 )/< |[ 32 ]| >
gap> M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> _M := LeftPresentation( [ 2^3 ], R );
<A cyclic left module presented by an unknown number of relations for a cyclic\
 generator>
gap> Display( _M );
Z/( 256 )/< |[ 8 ]| >
gap> _M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> alpha2 := HomalgMap( [ 1 ], M, _M );
<A "homomorphism" of left modules>
gap> IsMorphism( alpha2 );
true
gap> alpha2;
<A homomorphism of left modules>
 gap> Display( alpha2 );
 [ [  1 ] ]

 modulo [ 256 ]

 the map is currently represented by the above 1 x 1 matrix
gap> M_ := Kernel( alpha2 );
<A cyclic left module presented by yet unknown relations for a cyclic generato\
r>
gap> alpha1 := KernelEmb( alpha2 );
<A monomorphism of left modules>
gap> seq := HomalgComplex( alpha2 );
<An acyclic complex containing a single morphism of left modules at degrees
[ 0 .. 1 ]>
gap> Add( seq, alpha1 );
gap> seq;
<A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> IsShortExactSequence( seq );
true
gap> seq;
<A short exact sequence containing 2 morphisms of left modules at degrees
[ 0 .. 2 ]>
```

```
 gap> Display( seq );
 -------------------------
 at homology degree: 2
 Z/( 256 )/< |[ 4 ]| >
 -------------------------
 [ [  24 ] ]

 modulo [ 256 ]

 the map is currently represented by the above 1 x 1 matrix
 ------------v------------
 at homology degree: 1
 Z/( 256 )/< |[ 32 ]| >
 -------------------------
 [ [  1 ] ]

 modulo [ 256 ]

 the map is currently represented by the above 1 x 1 matrix
 ------------v------------
 at homology degree: 0
 Z/( 256 )/< |[ 8 ]| >
 -------------------------
gap> K := LeftPresentation( [ 2^7 ], R );
<A cyclic left module presented by an unknown number of relations for a cyclic\
 generator>
gap> L := RightPresentation( [ 2^4 ], R );
<A cyclic right module on a cyclic generator satisfying an unknown number of r\
elations>
gap> triangle := LHomHom( 4, seq, K, L, "t" );
<An exact triangle containing 3 morphisms of left complexes at degrees
[ 1, 2, 3, 1 ]>
gap> lehs := LongSequence( triangle );
<A sequence containing 14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> ByASmallerPresentation( lehs );
<A non-zero sequence containing 14 morphisms of left modules at degrees
[ 0 .. 14 ]>
gap> IsExactSequence( lehs );
false
gap> lehs;
<A non-zero left acyclic complex containing
14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> Assert( 0, IsLeftAcyclic( lehs ) );
 gap> Display( lehs );
 -------------------------
 at homology degree: 14
 Z/( 256 )/< |[ 4 ]| >
 -------------------------
 [ [  4 ] ]

 modulo [ 256 ]

 the map is currently represented by the above 1 x 1 matrix
```

```
------------v------------
at homology degree: 13
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  6 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 12
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 11
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 10
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  6 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 9
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 8
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]
```

```
the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 7
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  6 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 6
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 5
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 4
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  6 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 3
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  8 ] ]

modulo [ 256 ]
```

```
the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 16 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
Z/( 256 )/< |[ 8 ]| >
-------------------------
```

3.2 Betti Diagrams

3.2.1 Schenck-3.2

This is an example from Section 3.2 in [Sch03].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> mmat := HomalgMatrix( "[ x, x^3 + y^3 + z^3 ]", 1, 2, Qxyz );
<A homalg external 1 by 2 matrix>
gap> M := RightPresentationWithDegrees( mmat );
<A graded cyclic right module on a cyclic generator satisfying 2 relations>
gap> Mr := Resolution( M );
<A right acyclic complex containing 2 morphisms of right modules at degrees
[ 0 .. 2 ]>
gap> bettiM := BettiDiagram( Mr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of right modules at degrees [ 0 .. 2 ]>>
gap> Display( bettiM );
 total:  1 2 1
--------------
     0:  1 1 .
     1:  . . .
     2:  . 1 1
--------------
degree:  0 1 2
gap> R := CoefficientsRing( Qxyz ) * "x,y,z,w";;
gap> nmat := HomalgMatrix( "[ z^2 - y*w, y*z - x*w, y^2 - x*z ]", 1, 3, R );
<A homalg external 1 by 3 matrix>
gap> N := RightPresentationWithDegrees( nmat );
<A graded cyclic right module on a cyclic generator satisfying 3 relations>
gap> Nr := Resolution( N );
<A right acyclic complex containing 2 morphisms of right modules at degrees
[ 0 .. 2 ]>
gap> bettiN := BettiDiagram( Nr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of right modules at degrees [ 0 .. 2 ]>>
```

```
gap> Display( bettiN );
 total:  1 3 2
--------------
     0:  1 . .
     1:  . 3 2
--------------
degree:  0 1 2
```

3.2.2 Schenck-8.3

This is an example from Section 8.3 in [Sch03].

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,w";;
gap> jmat := HomalgMatrix( "[ z*w, x*w, y*z, x*y, x^3*z - x*z^3 ]", 1, 5, R );
<A homalg external 1 by 5 matrix>
gap> J := RightPresentationWithDegrees( jmat );
<A graded cyclic right module on a cyclic generator satisfying 5 relations>
gap> Jr := Resolution( J );
<A right acyclic complex containing 3 morphisms of right modules at degrees
[ 0 .. 3 ]>
gap> betti := BettiDiagram( Jr );
<A Betti diagram of <A right acyclic complex containing
3 morphisms of right modules at degrees [ 0 .. 3 ]>>
gap> Display( betti );
 total:  1 5 6 2
----------------
     0:  1 . . .
     1:  . 4 4 1
     2:  . . . .
     3:  . 1 2 1
----------------
degree:  0 1 2 3
```

3.2.3 Schenck-8.3.3

This is Exercise 8.3.3 in [Sch03].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> mat := HomalgMatrix( "[ x*y*z, x*y^2, x^2*z, x^2*y, x^3 ]", 1, 5, Qxyz );
<A homalg external 1 by 5 matrix>
gap> M := RightPresentationWithDegrees( mat );
<A graded cyclic right module on a cyclic generator satisfying 5 relations>
gap> Mr := Resolution( M );
<A right acyclic complex containing 3 morphisms of right modules at degrees
[ 0 .. 3 ]>
gap> betti := BettiDiagram( Mr );
<A Betti diagram of <A right acyclic complex containing
3 morphisms of right modules at degrees [ 0 .. 3 ]>>
gap> Display( betti );
 total:  1 5 6 2
----------------
     0:  1 . . .
```

```
     1:  . . . .
     2:  . 5 6 2
----------------
degree:  0 1 2 3
```

3.3 Commutative Algebra

3.3.1 Saturate

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
<A graded (left) ideal given by 3 generators>
gap> m := GradedLeftSubmodule( "x,y,z", R );
<A graded (left) ideal given by 3 generators>
gap> J := Intersect( m^3, GradedLeftSubmodule( "x", R ) );
<A graded (left) ideal given by 6 generators>
gap> Jm := SubmoduleQuotient( J, m );
<A graded (left) ideal given by 3 generators>
gap> J_m := Saturate( J, m );
<A graded principal (left) ideal given by a cyclic generator>
gap> Js := Saturate( J );
<A graded principal (left) ideal given by a cyclic generator>
gap> Assert( 0, Js = J_m );
```

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Index